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Recurrence and Return Times of the Sierpinski Carpet

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We compute the limit distribution of the recurrence and of the normalized kth return times to small sets of the Sierpinski carpet with respect to a natural measure defined on it. It is proved that this dynamical system follows the Poisson law, as one could have expected for such schemes. We study different sequences which converge in finite distribution to the Poisson point process. This limit in law is very interesting in ergodic theory, and it seems to appear for chaotic dynamical systems such as the one we study.

KEY WORDS: Fractal; thermodynamic formalism; recurrence; return times; Poisson law.

1. INTRODUCTION

We study the recurrence and the return times of orbits of points of a dynamical system, the *Sierpinski carpet*. In probability theory, one may obtain at the limit in distribution of some processes normal laws or Poisson laws. Recent works by Sinaï[7,8] for the quantom kicked rotator model and by Hirata[2] for Axiom A diffeomorphisms show that the *Poisson law property* holds. It seems that one could expect to obtain similar results for chaotic dynamical systems (Bernoulli, mixing,...) which are very interesting in ergodic theory. Here we prove that the Poisson law property holds almost everywhere with respect to a natural measure defined on the Sierpinski carpet. See also ref. 5 for analogous results.

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2. THE MODEL

We prove the Poisson law limit for the Sierpinski carpets equipped with natural measures. Sierpinski carpets[4,6] are fractal planar sets $\bar{S}$ which are generalized Cantor sets. Given integers $n \geqslant m$ and a set

$$S \subset \{(i, j)/0 \leqslant i < n \text{ and } 0 \leqslant j < m\}$$

with $\#(S) = p$, we define in dimension 2 the fractal set $\bar{S}$ by

$$\bar{S} = \left\{ \left(\sum_{k \geqslant 1} \frac{x_k}{n^k}, \sum_{k \geqslant 1} \frac{y_k}{m^k} \right) \Big/ \forall k, (x_k, y_k) \in S \right\} \tag{1}$$

We define then a natural measure μ on the square $[0, 1]^2$ by giving values only to the rectangles defining S: the rectangles V_k have the measures

$$\mu(V_k) = \theta_k \tag{2}$$

as measured by μ (the other rectangles have measure 0). By refining this method, we construct a measure μ on $\bar{S}$.

Actually, if $(f_k)_{k=1,\dots,p}$ represent affine maps contracting by a factor of n horizontally and m vertically, i.e., for k which corresponds to a pair $(i, j) \in S$,

$$f_k(x, y) = \left(\frac{i + x}{n}, \frac{j + y}{m} \right) \qquad \text{for} \quad (x, y) \in [0, 1]^2 \qquad (\text{and } f_{|V_k} := f_k)$$

(and $f \equiv 0$ on the other rectangles), we have then

$$\bar{S} = \bigcup_{k=1}^{p} f_k(\bar{S}) = \left\{ \bigcap_{j \geqslant 1} \bigcup_{\{i_1,\dots,i_j\}} f_{i_1} f_{i_2} \cdots f_{i_j}([0, 1]^2) \right\}$$

and μ is invariant with respect to f and the f_k. We also define a map

$$\begin{aligned} \varphi: \quad S_p = \{1, 2, \dots, p\}^{\mathbb{N}^*} &\to \bar{S} \\ (i_1, i_2, \dots) &\to \bigcap_{j \geqslant 1} f_{i_1} f_{i_2} \cdots f_{i_j}([0, 1]^2) \end{aligned} \tag{3}$$

which is a surjection; moreover, it is bounded-to-one and one-to-one on a set of Lebesgue measure 1 and of μ measure 1. The measure μ satisfies

$$\mu[\varphi(i_1, i_2, \dots, i_k)] = \rho(i_1, i_2, \dots, i_k) = \prod_{j=1}^{k} \mu(V_{i_j}) = \prod_{j=1}^{k} \theta_{i_j} \tag{4}$$

where the measure ρ is defined on symbolics. By the Kolmogorov consistency theorem, we take μ to be the unique measure on the Borel subsets of $\bar{S}$ satisfying (4). We introduce the cylinders $C(n, \underline{y})$ for $n \geqslant 1$ on S_p

$$C(n, \underline{y}) = \{\underline{x} \in S_p / x_1 = y_1; ...; x_n = y_n\}$$

and the shift σ by $\sigma(\underline{x}) = \underline{y} \Leftrightarrow \forall n \in \mathbb{N}^*,\ y_n = x_{n+1}$.

Obviously the map φ assures a correspondance between the two dynamical systems $(\bar{S}, \mu, f)$ and (S_p, ρ, σ) (thermodynamic formalism[1, 2]), i.e.,

$$\mu = \varphi^* \rho \qquad \text{and} \qquad \forall n \in \mathbb{N},\ f^n \circ \varphi = \varphi \circ \sigma^n$$

Following ref. 2, we define for a point $y \in \bar{S}$ and an ε-neighborhood $U_\varepsilon(y)$

$$\mu_\varepsilon = \frac{\mu_{|U_\varepsilon(y)}}{\mu(U_\varepsilon(y))}$$

which is the induced measure and the kth return of a point x from $U_\varepsilon(y)$ to $U_\varepsilon(y)$ by $T^{(k)}_{\varepsilon, y}(x)$. We introduce a counting measure on $\mathbb{R}^+$ by

$$Y_\varepsilon(y) = \sum_{k \geqslant 1} \delta_{T^{(k)}_{\varepsilon, y}(x) / E_\mu[T^{(1)}_{\varepsilon, y}(x)]} \tag{5}$$

and for $B \in \mathscr{B}(\mathbb{R}^+)$, $Y_\varepsilon(B)$ is the number of times that the normalized return times lie in B. In many situations it turns out that the limit distribution is actually a Poisson point process, and we prove μ-a.e. this *Poisson law property* for the Sierpinski carpet.

We introduce the recurrence to small sets, which is the amount of time that the orbits of a point $x \in U_\varepsilon(y)$ spend in $U_\varepsilon(y)$. To this purpose, let

$$W_\varepsilon(y) = \sum_{j=0}^{[\alpha_\varepsilon]-1} 1_{\{U_\varepsilon(y)\}} \circ f^j \qquad \text{where} \quad \alpha_\varepsilon = \frac{1}{E_\mu[T^{(1)}_{\varepsilon, y}(x)]} \tag{6}$$

Our main result is that the random variables $Y_\varepsilon(y)$ and $W_\varepsilon(y)$ tend for μ-a.e. $y \in \bar{S}$ respectively to the Poisson point process and the Poisson law, and the result does not hold for any point: for example, the ones which have a periodic symbolic expansion. This is given in the following results.

Theorem 1. The random variables $W_\varepsilon[y]$ converge for μ-a.e. $y \in \bar{S}$ when ε goes to 0 to the Poisson law $\mathscr{P}(1)$.

Theorem 2. For μ-a.e. $y \in \bar{S}$ the limit distribution of the random variables $Y_\varepsilon(y)$ is the Poisson point process, i.e., for any disjoint Borel sets $B_1, B_2, ..., B_q \in \mathscr{B}(\mathbb{R}^+)$ and any nonnegative integers $k_1, ..., k_q$ we get

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(Y_\varepsilon[y](B_1) = k_1; ...; Y_\varepsilon[y](B_q) = k_q) = \prod_{i=1}^{q} \frac{\lambda(B_i)^{k_i}}{k_i!} e^{-\lambda(B_i)}$$

Remarks. 1. The results are also valid for dynamical systems equipped with Markovian measures and associated to symbolics which are Markov chains.

2. We prove that periodic points do not satisfy Theorem 1. Hirata proved[2] that for these points the limit distribution of the normalized first return time is a combination of the delta distribution and the exponential distribution.

3. What happens if we take shift-invariant measures which can be approximated weakly and in entropy by Markov measures (ref. 3, Appendix 1)?

We have a similar result for the symbolic dynamical system (S_p, ρ, σ), which is strongly mixing (and therefore ergodic) since it is a Bernoulli shift. It has then a chaotic behavior and has the *Poisson law property*. The small sets $U_\varepsilon(y)$ are replaced for $\underline{y} \in S_p$ by small cylinders $C(n, \underline{y})$; the return times for $\underline{x} \in C(n, \underline{y})$ are

$$T_{n,\underline{y}}^{(k+1)}(\underline{x}) = \inf\{i > T_{n,\underline{y}}^{(k)}(\underline{x}) / \sigma^i(\underline{x}) \in C(n, \underline{y})\} \qquad \text{where} \quad T_{n,\underline{y}}^{(0)}(\underline{x}) = 0 \quad (7)$$

Let $(W_n(\underline{y}))_{n \geq 1}$ be the sequence of random variables defined for any $\underline{y} \in S_p$ by

$$W_n(\underline{y}) = \sum_{j=0}^{[\alpha_n]-1} 1_{\{C(n,\underline{y})\}} \circ \sigma^j \qquad \text{where} \quad \alpha_n = E_{\rho_n}[T_{n,\underline{y}}^{(1)}] = \frac{1}{\rho(C(n, \underline{y}))} \quad (8)$$

The induced measures ρ_n satisfy

$$\rho_n = \frac{\rho_{|C(n,\underline{y})}}{\rho(C(n, \underline{y}))}$$

We get then the following result.

Theorem 3. The sequence $(W_n(\underline{y}))_{n \geq 1}$ converges in distribution when n goes to $+\infty$ for ρ-a.e. point $\underline{y} \in S_p$ to the Poisson law $\mathscr{P}(1)$.

Remarks. 1. If we take any real $\lambda > 0$, we get for ρ-a.e. $\underline{y} \in S_p$

$$\zeta_n(\underline{y}) = \sum_{j=0}^{[\lambda\alpha_n]-1} 1_{\{C(n,\underline{y})\}} \circ \sigma^j \xrightarrow[n \to +\infty]{\mathscr{L}} \mathscr{P}(\lambda) \quad (9)$$

2. We may replace λ by $\lambda_n \to 0$ such that $\lambda_n \rho(C(n, \underline{y})) \to +\infty$, and we get

$$\xi_n(\underline{y}) = \sum_{j=0}^{[\lambda_n \alpha_n]-1} 1_{\{C(n, \underline{y})\}} \circ \sigma^j \approx \mathscr{P}(\lambda_n)$$

i.e.,

$$\forall k \in \mathbb{N}, \quad \rho(\xi_n(\underline{y}) = k) \sim e^{-\lambda_n} \frac{(\lambda_n)^k}{k!} \tag{10}$$

Similarly to Theorem 2, we prove also the following result.

Theorem 4. For any real $t > 0$ the sequence $(N_n[\underline{y}]([0; t]))_{n \geqslant 1}$, where

$$N_n[\underline{y}]([0; t]) = \sum_{k \geqslant 1} 1_{\{\rho(C(n, y))\, T^{(k)}_{n, y} \leqslant t\}}$$

converges in distribution for ρ-a.e. $\underline{y} \in S_p$ to the Poisson law $\mathscr{P}(t)$.

Remark. We can easily generalize in the following way: for ρ-a.e. $\underline{y} \in S_p$ and for any disjoint Borel sets $B_1, B_2, \ldots, B_q \in \mathscr{B}(\mathbb{R}^+)$ and for any nonnegative integers $k_1, k_2, \ldots, k_q$ we get

$$\lim_{n \to +\infty} \rho_n(N_n[\underline{y}](B_1) = k_1; \ldots; N_n[\underline{y}](B_q) = k_q) = \prod_{i=1}^{q} \frac{\lambda(B_i)^{k_i}}{k_i!} e^{-\lambda(B_i)} \tag{11}$$

which means that the limits in distribution of the $N_n[\underline{y}](B_i)$ are independant Poisson laws of parameters $\lambda(B_i)$, the Lebesgue measure of B_i.

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